MATHEMATICS

ON THE ALGEBRAIC APPROXIMATION OF FUNCTIONS. IV

BY

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To K. MAHLER

In this paper, we follow, unless stated to the contrary, the notation and terminology of the previous papers in this series [1]. However, we consider only a particular case of the general theory given in (1). Namely, we take the constant field F to be the field C of complex numbers, we take ω to be the ring $C[z]$ of polynomials with complex coefficients, and we take the valuation $\lceil\ \rceil$ to be the degree of a polynomial. Further, we let G be any connected open subset of C, and Π: $z_1, z_2, \dots$ an arbitrary infinite sequence of equal or distinct points of G. Then we can take the ring of analytic functions on G to be a weld ring of the sequence of primes Π: $z-z_1, z-z_2, \dots$. Of course, this simply means that the general theory of [1] can be applied to any vector $f_1(z), \dots, f_m(z)$ of analytic functions on G, m being any positive integer.

Throughout this paper, we shall abuse language and speak of the approximation of a vector of analytic functions on G with respect to the sequence of *points* Π: $z_1, z_2, \dots$, when, in the terminology of [1], we mean approximation with respect to the sequence of *primes* Π: $z-z_1, z-z_2, \dots$.

In the first part of this paper, we shall prove a very general theorem which asserts that a vector $f_1(z), \dots, f_m(z)$ of analytic functions on G, which is linearly independent over the field of rational functions, is perfect with respect to "most" sequences of points Π in G. This theorem is remarkable, since previously only a few examples of perfect function vectors were known (see § 23 of [1]). However, the theorem is purely an existence one, and unfortunately gives no method for explicitly constructing the Latin and German approximation polynomials. The explicit construction of these approximation polynomials seems to be a difficult problem.

In the second part, we consider the following general question. If $f_1(z), \dots, f_m(z)$ is any vector of analytic functions which is perfect with respect to any sequence Π: $z_1, z_2, \dots$ of distinct points of G, we determine a lower bound for

$$\max_{z=z_1,\dots,z_{m\varrho}} |p_1(z)f_1(z)+\dots+p_m(z)f_m(z)|$$

where $p_1(z), \ldots, p_m(z)$ are arbitrary polynomials which are not all identically zero, and $\varrho - 1 = \max_k \overline{|p_k(z)|}$. This lower bound is in terms of

$$(*) \qquad \max_{h,k} \max_{z=z_1,\ldots,z_{m\varrho}} |\mathfrak{A}_{hk}(z|\varrho \ldots \varrho)| \qquad (h, k=1, \ldots, m),$$

where

$$\mathfrak{A}_{hk}(z|\varrho \ldots \varrho) \qquad (h, k=1, \ldots, m)$$

are the unique German approximation polynomials defined in § 11 of [1]. I have been unable, in general, to give an explicit upper estimate for (*), although it does not seem unreasonable to do so. This general theorem is therefore unsatisfactory in its present form.

However, for the particular function vector $\alpha_1^z, \ldots, \alpha_m^z$, $\alpha_1, \ldots, \alpha_m$ being distinct non-zero complex numbers, and the particular sequence Π_1: 0, 1, 2, ..., the approximation polynomials

$$\mathfrak{A}_{hk}(z|\varrho \ldots \varrho) \qquad (h, k=1, \ldots, m)$$

were explicitly constructed in § 26 of [1]. Using these explicit expressions, one can give an explicit upper estimate for (*), and the general theorem then yields a result analogous to that of Turán and Dancs [2].

In the light of the applications Turán and Dancs give for their results [2], [3], it would be very interesting to obtain an explicit upper estimate for (*) in the general case.

I wish to express my thanks to A. van der Poorten for stimulating discussions on this work.

1. The general perfectness theorem is as follows.

Theorem 1. *Let $f_1(z), \ldots, f_m(z)$ be any vector of analytic functions on G, which is linearly independent over the field of rational functions, and let S be an arbitrary finite or countably infinite subset of G. Then there exist infinitely many sequences Π: $z_1, z_2, \ldots$ of distinct points of G, with respect to which this function vector is perfect, and which contain a subsequence converging to each point of S.*

Proof. The set S being given, we prove by a recursive argument that there exist infinitely many sequences Π: $z_1, z_2, \ldots$ satisfying the assertions of the theorem.

A sequence Π: $z_1, z_2, \ldots$ satisfying the assertions of the theorem is defined as follows.

Firstly, choose for z_1 any point in G at which none of the functions $f_1(z), \ldots, f_m(z)$ vanish. Such a choice is always possible since the zeros of these functions form a discrete subset of G. By this choice of z_1, it follows from Criterion 2 of § 13 of [1] [1]), that the function vector $f_1(z), \ldots, f_m(z)$ is normal at all systems of parameters $\varrho_1, \ldots, \varrho_m$ with sum $\sigma \leqslant 1$.

Next, let Σ be any positive integer, and suppose that we have constructed the points $z_1, \ldots, z_\Sigma$ in such a fashion that $f_1(z), \ldots, f_m(z)$ is normal at all systems of parameters $\varrho_1, \ldots, \varrho_m$ with sum $\sigma \leqslant \Sigma$. We show that we can choose the point $z_{\Sigma+1}$ in any sufficiently small neighbourhood of any point in S such that $f_1(z), \ldots, f_m(z)$ is normal at all systems of parameters $\varrho_1, \ldots, \varrho_m$ with sum $\sigma \leqslant \Sigma+1$.

To see this, let $\{(\varrho_1^{(n)}, \ldots, \varrho_m^{(n)}) \,|\, n=1, \ldots, N(\Sigma)\}$ be the set of all systems of parameters with sum equal to Σ. Then, since $f_1(z), \ldots, f_m(z)$ is normal at all such systems of parameters $\varrho_1^{(n)}, \ldots, \varrho_m^{(n)}$, the Second Uniqueness Theorem of § 11 of (1) implies that the Latin remainder functions

$$r_h(z \,|\, \varrho_1^{(n)} \ldots \varrho_m^{(n)}) \qquad (h=1, \ldots, m;\, n=1, \ldots, N(\Sigma))$$

are *uniquely* determined, except for a constant factor, by the points $z_1, \ldots, z_\Sigma$. Further, since the function vector $f_1(z), \ldots, f_m(z)$ is linearly independent over the field of rational functions, none of these remainder functions is identically zero, and thus their zeros form a *discrete* subset of G. We can therefore choose $z_{\Sigma+1}$ in any sufficiently small neighbourhood of any point of S such that none of

$$r_h(z \,|\, \varrho_1^{(n)} \ldots \varrho_m^{(n)}) \qquad (h=1, \ldots, m;\, n=1, \ldots, N(\Sigma))$$

vanish at $z_{\Sigma+1}$, i.e.

$$\underline{|\, r_h(z \,|\, \varrho_1^{(n)} \ldots \varrho_m^{(n)}) \,|} = \Sigma \qquad (h=1, \ldots, m;\, n=1, \ldots, N(\Sigma)).$$

But, since these remainder functions are *unique*, it follows from Criterion 2 of § 13 of [1] [1]) that $f_1(z), \ldots, f_m(z)$ is normal at all systems of parameters

$$\varrho_1^{(n)} + \delta_{h1}, \ldots, \varrho_m^{(n)} + \delta_{hm} \qquad (h=1, \ldots, m;\, n=1, \ldots, N(\Sigma)),$$

i.e. $f_1(z), \ldots, f_m(z)$ is normal at all systems of parameters $\varrho_1, \ldots, \varrho_m$ with sum $\sigma \leqslant \Sigma+1$.

Thus the sequence Π: $z_1, z_2, \ldots$ is recursively defined, and it obviously satisfies the assertions of the theorem.

Further, it is clear that the above argument proves the existence of infinitely many sequences satisfying the assertions of the theorem. This completes the proof.

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[1]) As Professor Popken has pointed out to me, we use here a slightly weakened form of normality and consequently a slightly weakened form of Criterion 2 of § 13 of [1]. The slightly weakened form of normality at $\varrho_1, \ldots, \varrho_m$ is as follows. Instead of insisting that the function vector $f_1(z), \ldots, f_m(z)$ vanishes at none of the points $\Pi : z_1, z_2, \ldots$, we insist only that it vanishes at none of the points $z_1, \ldots, z_\sigma$. The rest of the definition remains unchanged. The whole theory of normality given in [1] obviously holds for this slightly weakened form of normality, and, of course, the notion of $f_1(z), \ldots, f_m(z)$ being *perfect* remains the same.

As mentioned before, Theorem 1 is an existence theorem, and gives no method for *explicitly* constructing the approximation polynomials. Indeed, it does not even give *explicitly* the sequences Π: $z_1, z_2, \ldots$ satisfying the assertions of the theorem.

It is also worth noting that the proof of Theorem 1 shows that there exist infinitely many sequences sequences Π: $z_1, z_2, \ldots$, which, in addition to satisfying the assertions of the theorem, are contained in any given *dense* subset of G.

2. Let $f_1(z), \ldots, f_m(z)$ be any vector of analytic functions on G, which is perfect with respect to a sequence Π: $z_1, z_2, \ldots$ of distinct points of G. If $p_1(z), \ldots, p_m(z)$ are arbitrary polynomials, which are not all identically zero, and $\varrho - 1 = \max\limits_k \overline{|p_k(z)|}$, we now determine a lower bound for

$$\max_{z=z_1,\ldots,z_{m\varrho}} |p_1(z) f_1(z) + \ldots + p_m(z) f_m(z)|.$$

If $f(z)$ is any analytic function on G, let us put $\|f(z)\| = \max\limits_{z=z_1,\ldots,z_{m\varrho}} |f(z)|$.

Define $F(z) = p_1(z) f_1(z) + \ldots + p_m(z) f_m(z)$. Then the identity

$$\mathfrak{A}_{hj}(z|\varrho \ldots \varrho) F(z) = f_j(z) \sum_{k=1}^{m} p_k(z) \mathfrak{A}_{hk}(z|\varrho \ldots \varrho) + \\ + \sum_{k=1}^{m} p_k(z) \mathfrak{R}_{hkj}(z|\varrho \ldots \varrho) \qquad (h, j = 1, \ldots, m)$$

obviously holds, where

$$\mathfrak{A}_{hk}(z|\varrho \ldots \varrho),\ \mathfrak{R}_{hkj}(z|\varrho \ldots \varrho) \qquad (h, k, j = 1, \ldots, m)$$

are the unique German polynomials and remainder functions defined in § 11 of [1]. Since all of the remainder functions

$$\mathfrak{R}_{hkj}(z|\varrho \ldots \varrho) \qquad (h, k, j = 1, \ldots, m)$$

vanish at all the points $z_1, \ldots, z_{m\varrho}$, it follows immediately that

$$\|\mathfrak{A}_{hj}(z|\varrho \ldots \varrho) F(z)\| = \|f_j(z) \sum_{k=1}^{m} p_k(z) \mathfrak{A}_{hk}(z|\varrho \ldots \varrho)\| \qquad (h, j = 1, \ldots, m),$$

whence

$$\|F(z)\| \geqslant \frac{\|f_j(z) \sum_{k=1}^{m} p_k(z) \mathfrak{A}_{hk}(z|\varrho \ldots \varrho)\|}{\|\mathfrak{A}_{hj}(z|\varrho \ldots \varrho)\|} \qquad (h, j = 1, \ldots, m). \tag{1}$$

Now choose the index j so that $\min\limits_{z=z_1,\ldots,z_{m\varrho}} |f_j(z)| = \max\limits_k \min\limits_{z=z_1,\ldots,z_{m\varrho}} |f_k(z)|$, and the index h so that $p_h(z)$ is any of those polynomials among $p_1(z), \ldots, p_m(z)$ having degree equal to $\varrho - 1$. Let H be the highest coefficient of $p_h(z)$.

With this choice of the indices h and j, it follows from (1) that

$$\|F(z)\| \geqslant \max_k \min_{z=z_1,\ldots,z_{m\varrho}} |f_k(z)| \frac{\|\sum_{k=1}^{m} p_k(z) \mathfrak{A}_{hk}(z|\varrho \ldots \varrho)\|}{\|\mathfrak{A}_{hj}(z|\varrho \ldots \varrho)\|}. \tag{2}$$

An upper bound for $\|\mathfrak{A}_{hj}(z|\varrho \dots \varrho)\|$ is trivially given by

(3) $$\|\mathfrak{A}_{hj}(z|\varrho \dots \varrho)\| \leqslant \max_{h,k} \max_{z=z_1,\dots,z_{m\varrho}} |\mathfrak{A}_{hk}(z|\varrho \dots \varrho)|.$$

To obtain a lower bound for $\|\sum_{k=1}^{m} p_k(z)\mathfrak{A}_{hk}(z|\varrho \dots \varrho)\|$, we shall apply the following lemma.

Lemma. *If $b(z)$ is any polynomial of degree at most $m\varrho-1$ and highest coefficient β, then*

$$\|b(z)\| \geqslant |\beta|\,\mathfrak{M}(z_1, \dots, z_{m\varrho}),$$

where

$$\mathfrak{M}(z_1, \dots, z_{m\varrho}) = \left\{\sum_{i=1}^{m\varrho} \frac{1}{|z_i-z_1| \dots |z_i-z_{i-1}||z_i-z_{i+1}| \dots |z_i-z_{m\varrho}|}\right\}^{-1}.$$

Proof. By the Lagrange interpolation formula

$$b(z) = \sum_{i=1}^{m\varrho} b(z_i)\frac{(z-z_1)\dots(z-z_{i-1})(z-z_{i+1})\dots(z-z_{m\varrho})}{(z_i-z_1)\dots(z_i-z_{i-1})(z_i-z_{i+1})\dots(z_i-z_{m\varrho})}$$

and thus, equating highest coefficients,

$$\beta = \sum_{i=1}^{m\varrho} b(z_i)\frac{1}{(z_i-z_1)\dots(z_i-z_{i-1})(z_i-z_{i+1})\dots(z_i-z_{m\varrho})}.$$

Hence

$$|\beta| \leqslant \|b(z)\| \sum_{i=1}^{m\varrho} \frac{1}{|z_i-z_1| \dots |z_i-z_{i-1}||z_i-z_{i+1}| \dots |z_i-z_{m\varrho}|},$$

and the assertion of the lemma follows.

Now, since $f_1(z), \dots, f_m(z)$ is perfect with respect to Π: $z_1, z_2, \dots,$

$$\overline{|\mathfrak{A}_{hk}(z|\varrho \dots \varrho)|} = (m-1)\varrho - 1 + \delta_{hk} \qquad (k=1, \dots, m),$$

and, by definition, the highest coefficient of $\mathfrak{A}_{hh}(z|\varrho \dots \varrho)$ is 1 (see § 11 of [1]). It follows that

$$\sum_{k=1}^{m} p_k(z)\mathfrak{A}_{hk}(z|\varrho \dots \varrho) = Hz^{m\varrho-1} + \text{lower powers of } z.$$

Hence, applying the lemma, we obtain

(4) $$\|\sum_{k=1}^{m} p_k(z)\mathfrak{A}_{hk}(z|\varrho \dots \varrho)\| \geqslant |H|\,\mathfrak{M}(z_1, \dots, z_{m\varrho}).$$

Substituting the estimates (3) and (4) into (2), we arrive at the following theorem.

Theorem 2. *Let $f_1(z), \dots, f_m(z)$ be any vector of analytic functions on G, which is perfect with respect to a sequence Π: $z_1, z_2, \dots$ of distinct points of G. If $p_1(z), \dots, p_m(z)$ are arbitrary polynomials, which are not all identically zero, $\varrho - 1 = \max_k \overline{|p_k(z)|}$, and H is the highest coefficient of any of those* *polynomials among* $p_1(z), \ldots, p_m(z)$ *having degree equal to* $\varrho-1$, *then*

$$\frac{\max\limits_{z=z_1,\ldots,z_{m\varrho}} |p_1(z)f_1(z)+\ldots+p_m(z)f_m(z)|}{|H|} \geqslant$$

$$\geqslant \frac{\max\limits_{k} \min\limits_{z=z_1,\ldots,z_{m\varrho}} |f_k(z)|\,\mathfrak{M}(z_1,\ldots,z_{m\varrho})}{\max\limits_{h,k} \max\limits_{z=z_1,\ldots,z_{m\varrho}} |\mathfrak{A}_{hk}(z|\varrho\ldots\varrho)|},$$

where

$$\mathfrak{M}(z_1,\ldots,z_{m\varrho}) = \left\{\sum_{i=1}^{m\varrho} \frac{1}{|z_i-z_1|\ldots|z_i-z_{i-1}||z_i-z_{i+1}|\ldots|z_i-z_{m\varrho}|}\right\}^{-1},$$

and

$$\mathfrak{A}_{hk}(z|\varrho\ldots\varrho) \qquad (h, k=1,\ldots,m)$$

are the unique, normalized, German approximation polynomials of $f_1(z), \ldots, f_m(z)$ *with respect to the sequence* Π: $z_1, z_2, \ldots$ (*see* § 11 *of* [1] *for the precise definition*).

It seems remarkable that the lower bound for

$$\max_{z=z_1,\ldots,z_{m\varrho}} |p_1(z)f_1(z)+\ldots+p_m(z)f_m(z)|$$

given by Theorem 2 is *essentially independent of the coefficients of the arbitrary polynomials* $p_1(z), \ldots, p_m(z)$.

However, apart from this fact, Theorem 2 is unsatisfactory in its present form, since I can give no explicit upper bound for

$$\max_{h,k} \max_{z=z_1,\ldots,z_{m\varrho}} |\mathfrak{A}_{hk}(z|\varrho\ldots\varrho)|$$

for arbitrary function vectors $f_1(z), \ldots, f_m(z)$.

Finally, we apply Theorem 2 to the particular function vector $\alpha_1^z \ldots, \alpha_m^z$, $\alpha_1, \ldots, \alpha_m$ being distinct non-zero complex numbers, and the particular sequence Π_1: 0, 1, 2, ... For it was shown in § 24 of [1] that $\alpha_1^z, \ldots, \alpha_m^z$ is perfect with respect to Π_1: 0, 1, 2, ..., and further, in § 26 of [1], the German approximation polynomials were shown to be given explicitly by

$$\mathfrak{A}_{hk}(z|\varrho\ldots\varrho) = \frac{\alpha_h^{-\varrho-1}}{(\varrho-1)!}\prod_{\substack{l=1\\ l\neq h}}^{m}(\alpha_h-\alpha_l)^{-\varrho}\sum_{\lambda=0}^{\infty}(z-\lambda-1)\ldots(z-m\varrho+1)\alpha_k^{\lambda}\cdot$$

$$\cdot\left\{\frac{d^\lambda}{d\mathfrak{z}^\lambda}\prod_{l=1}^{m}(\mathfrak{z}-\alpha_l)^{\varrho-\delta_{hl}}\right\}_{\mathfrak{z}=\alpha_k}.$$

We now use these explicit expressions to obtain an explicit upper bound for $\max\limits_{h,k} \max\limits_{z=z_1,\ldots,z_{m\varrho}} |\mathfrak{A}_{hk}(z|\varrho\ldots\varrho)|$ in this particular case.

Put

$$\Delta = \max_k |\alpha_k|, \quad \omega = \min_k |\alpha_k|, \quad \delta = \min_{k\neq l} |\alpha_k-\alpha_l|.$$

Then clearly

$$\max_k \min_{z=0,\dots,m\varrho-1} |\alpha_k^z| = \min(1, \varDelta^{m\varrho-1}).$$

Further, $\prod_{l=1}^{m} (\mathfrak{z}-\alpha_l)^{\varrho-\delta_{hl}}$ is majorized by $(\mathfrak{z}+\varDelta)^{m\varrho-1}$, and thus

$$\left|\left\{\frac{d^\lambda}{d\mathfrak{z}^\lambda} \prod_{l=1}^{m} (\mathfrak{z}-\alpha_l)^{\varrho-\delta_{hl}}\right\}_{\mathfrak{z}=\alpha_k}\right| \leqslant (m\varrho-1)\dots(m\varrho-\lambda)(2\varDelta)^{m\varrho-1-\lambda}$$
$$(\lambda=0, 1, \dots, m\varrho-1).$$

Hence, if i is any of the points $0, 1, \dots, m\varrho-1$, it follows that

$$|\mathfrak{A}_{hk}(i|\varrho\dots\varrho)| \leqslant$$

$$\leqslant \frac{\omega^{-(\varrho-1)}}{(\varrho-1)!} \delta^{-(m-1)\varrho} \sum_{\lambda=i}^{m\varrho-1} (\lambda+1-i)\dots$$
$$\dots(m\varrho-1-i)\varDelta^\lambda(m\varrho-1)\dots(m\varrho-\lambda)(2\varDelta^{m\varrho-1-\lambda}),$$

$$\leqslant \frac{\omega^{-(\varrho-1)}}{(\varrho-1)!} \delta^{-(m-1)\varrho} (2\varDelta)^{m\varrho-1}(m\varrho-1)! \sum_{\lambda-i}^{m\varrho-1} \frac{(m\varrho-1-i)!}{(\lambda-i)!(m\varrho-\lambda-1)!} 2^{-\lambda},$$

$$\leqslant \frac{\omega^{-(\varrho-1)}}{(\varrho-1)!} \delta^{-(m-1)\varrho} (2\varDelta)^{m\varrho-1}(m\varrho-1)! \sum_{\lambda=0}^{m\varrho-1-i} \frac{(m\varrho-1-i)!}{\lambda!(m\varrho-1-i-\lambda)!} 2^{-\lambda-i},$$

$$\leqslant \frac{\omega^{-(\varrho-1)}}{(\varrho-1)!} \delta^{-(m-1)\varrho} (3\varDelta)^{m\varrho-1}(m\varrho-1)!,$$

i.e.

(5) $$\max_{h,k} \max_{z=0,\dots,m\varrho-1} |\mathfrak{A}_{hk}(z|\varrho\dots\varrho)| \leqslant \frac{\omega^{-(\varrho-1)}}{(\varrho-1)!} \delta^{-(m-1)\varrho}(3\varDelta)^{m\varrho-1}(m\varrho-1)!.$$

Finally, we observe that

(6) $$\mathfrak{M}(0, 1, \dots, m\varrho-1) = \left\{\sum_{i=0}^{m\varrho=1} \frac{1}{i!(m\varrho-1-i)!}\right\}^{-1} = \frac{(m\varrho-1)!}{2^{m\varrho-1}}.$$

Substituting the estimates (5) and (6) into Theorem 2, we obtain the following result.

Theorem 3. *Let $\alpha_1, \dots, \alpha_m$ be any distinct non-zero complex numbers, $p_1(z), \dots, p_m(z)$ arbitrary polynomials which are not all identically zero, $\varrho-1 = \max_k \overline{|p_k(z)|}$, and H the highest coefficient of any of those polynomials among $p_1(z), \dots, p_m(z)$ having degree equal to $\varrho-1$. Put*

$$\varDelta = \max_k |\alpha_k| \,, \quad \omega = \min_k |\alpha_k| \,, \quad \delta = \min_{k\neq l} |\alpha_k-\alpha_l|.$$

Then

$$\frac{\max_{z=0,\dots,m\varrho-1} |p_1(z)\alpha_1^z+\dots+p_m(z)\alpha_m^z|}{|H|} \geqslant \min(1, \varDelta^{m\varrho-1}) \frac{(\varrho-1)!\,\omega^{\varrho-1}\delta^{(m-1)\varrho}}{(6\varDelta)^{m\varrho-1}}.$$

Turán and Dancs [2] have obtained an analogous result to Theorem 3, and they have applied their results to study the distribution of the zeros of certain classes of analytic functions. I have not investigated the application of Theorems 2 and 3 to such questions.

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